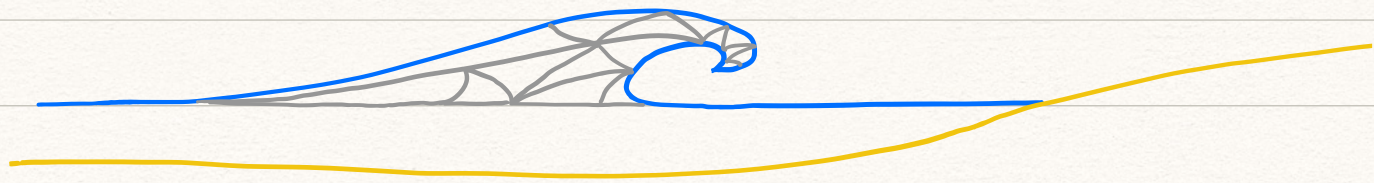
How to Solve Quantum Gravity *★1‡

* In 2 dimensions
★ In Euclidean signature
1 For orientable manifolds
‡ With no boundaries



0. Contents

1. Introduction

Faust: "Quantum gravity is easy! Look:

$$Z = \int Dg \, \exp\left[-\frac{1}{16\pi G}\int d^2x \sqrt{g}\, R\right] \overset{[GB]}{=}$$

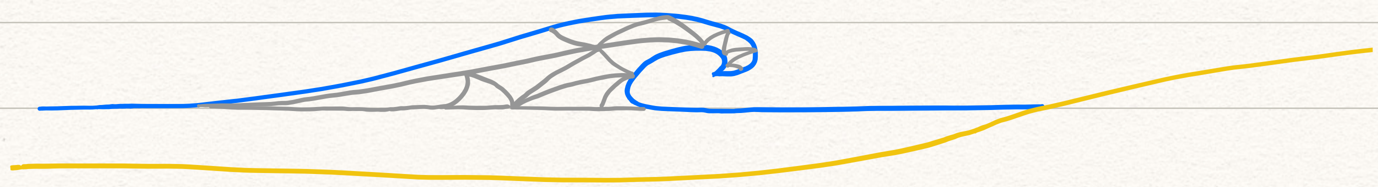
$$= \int Dg \, e^{-\chi/4G} = \sum_{h=0}^{\infty} \int Dg_h \, e^{\frac{h-1}{2G}} =$$

$\chi = 2-2h$

$$= e^{-\frac{1}{2G}} \sum_{h=0}^{\infty} e^{\frac{h}{2G}} \, \mathrm{vol}\left(\text{all metrics on 2D surfaces of genus } h\right).$$

... Hmmm."

The Devil: "Well...



2. Moduli of Riemann Surfaces

2.1. Moduli Spaces

A Riemann surface Σ_g is an oriented 2D mfd with a complex (hence conformal) structure.

→ Goal: characterize & classify these surfaces.

Thm. (Uniformization). Topology ≡ geometry:

$$\tilde{\Sigma}_g = \begin{cases} \mathbb{P}, & g=0 \qquad R=+2 \quad \text{trivial} \\ \mathbb{C}, & g=1 \qquad R=0 \quad \text{classical} \\ \mathbb{H}, & g\geq 2 \qquad R=-2 \quad \text{hard} \end{cases}$$

//

Let $\hat{M}_g = \{\text{all RS of genus } g\}$, ∞-dimensional.
We want to study the moduli space

$$\mathcal{M}_g = \hat{M}_g / \mathrm{Diff}(\Sigma_g) = \{\text{all metrics on } \Sigma_g\}.$$

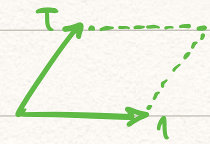
① study $T_g = \hat{M}_g / \mathrm{Diff}_0(\Sigma_g)$; [$\Sigma \sim \Sigma'$ if any biholomorphisms $\phi, \phi' : \Sigma \to \Sigma'$ are homotopic]

② mod out by $\Gamma_g = \mathrm{Diff}(\Sigma_g)/\mathrm{Diff}_0(\Sigma_g)$ (MCG).

Genus 0: $\mathbb{P}^1$ is the unique cpt RS in $g=0$.
Therefore $\mathcal{T}_0 = \mathcal{M}_0 = \{\mathbb{P}^1\}$.

Genus 1: T^2 is generated by $\tau_1, \tau_2 \in \mathbb{C}$, but...

- Dilations & rotations fix $\tau_1 = 1$
- $\tau \equiv \tau_2$ gives the same $T^{2'}$ as $\bar{\tau}$.

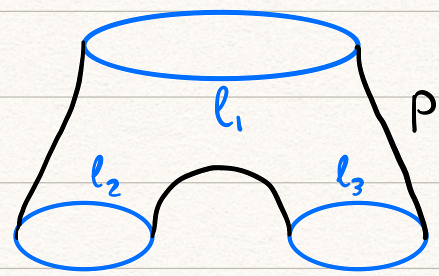


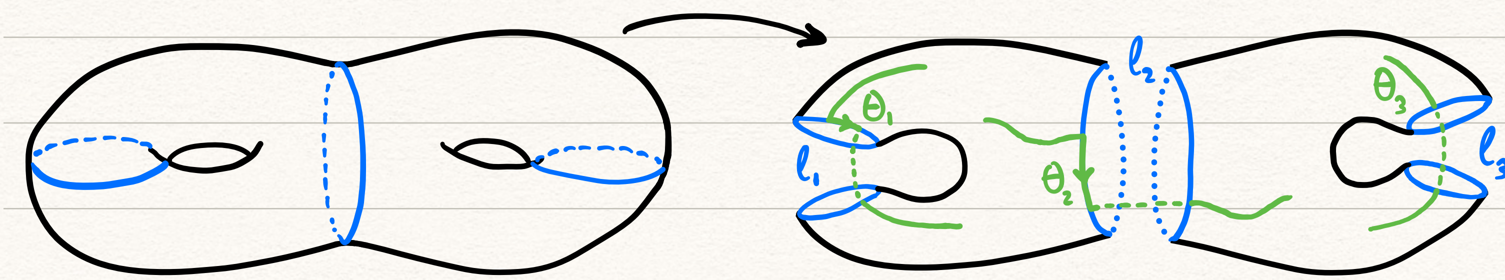
Hence $T^2 = \mathbb{C}/\langle 1, \tau\rangle$, $\tau \in \mathbb{H} \Longrightarrow \mathcal{T}_1 = \mathbb{H}$.

Thm. $\Gamma_1 = SL(2,\mathbb{Z})$. Thus $\mathcal{M}_1 = \mathbb{H}/\Gamma_1$ is 2-dim'l. //

Genus ≥2: decompose Σ_g into pairs of pants.
Each one has 3 bdy lengths $(\ell_1, \ell_2, \ell_3) \in \mathbb{R}^3_+$.

Thm. Each $\vec{\ell} = (\ell_1, \ell_2, \ell_3) \in \mathbb{R}^3_+$ uniquely specifies a hyperbolic structure on P. //





To glue together pants, we need:

- Lengths $\ell_i \in \mathbb{R}_+$ } for each of $3g-3$
- Twists $\theta_i \in \mathbb{R}$ } boundary curves.

Thus $\boxed{T_g = \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}}$ is $(6g-6)$-dimensional.

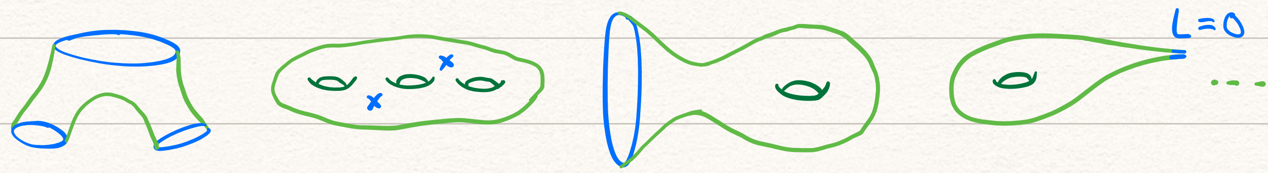
Thm. The $(\ell_i, \theta_i) \sim (q, p)$ are canonically conjugate & give T_g a symplectic form:

$$\omega_{WP} \equiv \sum_{i=1}^{3g-3} d\ell_i \wedge d\theta_i . \rightsquigarrow \text{Hamiltonian mechanics!}$$

2.2. Weil-Petersson Volumes

Add $n \in \mathbb{N}$ boundaries: everything still works! We restrict to ∂'s with fixed bdy lengths $\ell_i = L_i$.

$T_{g,n}(\vec{L}) = \mathbb{R}_+^{3g-3+n} \times \mathbb{R}^{3g-3+n}$ has dim. $6g-6+2n$.

Zoo: 

Finally, divide out Γ to obtain $\mathcal{M}_{g,n}(\vec{L})$. Compactify to $\overline{\mathcal{M}}_{g,n}$, which allows pinches, etc.

Thm. $\overline{\mathcal{M}}_{g,n}(\vec{L}) \sim \mathbb{R}^{6g-6+2n}$ is:

- Simply connected & compact, but unorientable
- An orbifold finitely covered by a manifold
- Complex & Riemannian, inherited from $\mathcal{T}_{g,n}$
- Symplectic, with Weil-Petersson form

$$\boxed{\omega_{\vec{L}} = \sum_{i=1}^{3g-3+n} dl_i \wedge d\theta_i .}$$ [Sometimes $\omega_{\vec{L}} \longleftrightarrow \frac{1}{2}\omega_{\vec{L}}$] //

N.B. Topologically, $\overline{\mathcal{M}}_{g,n}(\vec{L}) = \overline{\mathcal{M}}_{g,n}(\vec{0}) \equiv \overline{\mathcal{M}}_{g,n}$.
However, $\omega_{\vec{L}} \neq \omega_{\vec{0}} \equiv \omega$: DH explains how.

The top form $\omega_{\vec{L}}^{3g-3+n}$ is a volume form.

Def. The Weil-Petersson volumes are

$$\boxed{V_{g,n}(\vec{L}) \equiv \frac{1}{(2\pi^2)^{3g-3+n}} \int_{\overline{\mathcal{M}}_{g,n}(\vec{L})} \frac{\omega_{\vec{L}}^{3g-3+n}}{(3g-3+n)!} = \frac{1}{(2\pi^2)^{3g-3+n}} \int_{\overline{\mathcal{M}}_{g,n}} e^{\omega_{\vec{L}}} .}$$ //

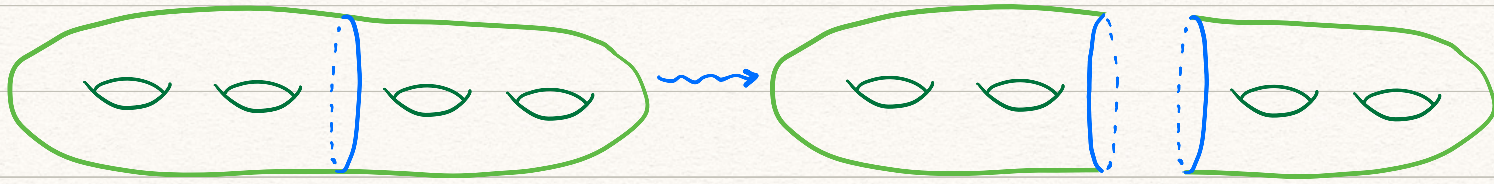
E.g. $V_{1,1}(0) = \frac{\pi^2}{12}$, $V_{1,1}(L) = \frac{\pi^2}{12} + \frac{L^2}{48}$, etc. //

2 ways to compute: ① recursion (McShane),
② symplectic reduction (DH).

2.3. Topological Recursion

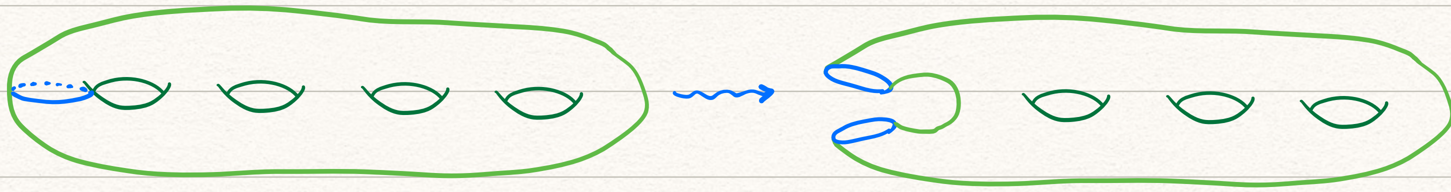
→ Idea: cut Σ_g along geodesics to get simpler moduli, then integrate over possible cuts.

- Separating curves: $\Sigma_g \rightsquigarrow \Sigma_{g_1,1}(b) \sqcup \Sigma_{g_2,1}(b)$



Hope for $V_g \overset{?}{=} \int_0^\infty db \int_0^b d\theta \, V_{g_1,b} V_{g_2,b}$.

- Non-separating curves: $\Sigma_g \rightsquigarrow \Sigma_{g-1,2}(b,b)$.



Hope for $V_g \overset{?}{=} \int_0^\infty db \int_0^b d\theta \, V_{g-1,2}(b,b)$.

- Boundaries $\vec{L}$ - partitioned arbitrarily:
  - separating: $\vec{L} \rightsquigarrow (\vec{L}_1, b) \sqcup (\vec{L}_2, b)$
  - non-sep: $\vec{L} \rightsquigarrow \vec{L} \sqcup (b,b)$

But $\nexists$ a modular-invariant choice of curve!
Suppose we had a "partition of unity" F, i.e.

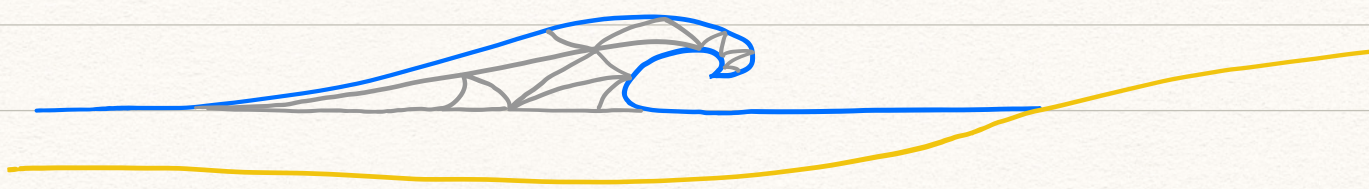
$\sum_\alpha F(b_\alpha) = 1$, $\alpha \in \{$closed geodesics on $\Sigma_g\}$. Then:

$$V_{g,n}(\vec{L}) \stackrel{?}{=} \int_0^\infty db \int_0^b d\theta \, F(b) \left[\boxed{S} + \boxed{NS} \right],$$

$$\boxed{S} = \frac{1}{2} \sum_{\substack{g_1, g_2 \\ g_1 + g_2 = g}} \sum_{\substack{\vec{L}_1, \vec{L}_2 \\ \vec{L}_1 \sqcup \vec{L}_2 = \vec{L}}} V_{g_1, n_1+1}(\vec{L}_1, b) \, V_{g_2, n_2+1}(\vec{L}_2, b),$$

$$\boxed{NS} = V_{g-1, n+2}(\vec{L}, b, b).$$

This captures the spirit, if not all the details, of MM's recursion.



3. Intersection Theory on $\overline{\mathcal{M}}_{g,n}(\vec{L})$

3.1. Intersection Numbers

How to get topological gravity correlators:

① Mark a pt $p_i \in \partial\Sigma_g$ on each bdy circle.

② Let $\widetilde{\mathcal{M}}_{g,n}(\vec{L})$ parametrize marked Σ_g.

$$\left[\widetilde{\mathcal{M}}_{g,n}(\vec{L}) \sim \overline{\mathcal{M}}_{g,n}(\vec{L}) \times \Sigma_g^n\right]$$

③ Let $(S^1)^n = T^n \circlearrowright \widetilde{\mathcal{M}}_{g,n}(\vec{L})$ rotate the p_i.

(π) The invariance of the L_i under S^1 leads to moment maps and symplectic reduction.

④ For each bdy of Σ_g, we get an S^1-bundle $C_i \longrightarrow \mathcal{M}_{g,n}(\vec{L})$. View this as a $U(1)$ theory.

⑤ Define $\psi_i \equiv c_1(C_i) \in H^2(\overline{\mathcal{M}}_{g,n}(\vec{L}), \mathbb{Q})$,

$$\tau_{i,d} \equiv \psi_i^d \in H^{2d}(\overline{\mathcal{M}}_{g,n}(\vec{L}), \mathbb{Q}),$$ and

$$\boxed{\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g \equiv \int_{\overline{\mathcal{M}}_{g,n}(\vec{L})} \tau_{i_1,d_1} \cdots \tau_{i_n,d_n} = \int_{\overline{\mathcal{M}}_{g,n}(\vec{L})} \psi_{i_1}^{d_1} \cdots \psi_{i_n}^{d_n} .}$$

These intersection numbers are only defined if

$$\dim \overline{\mathcal{M}}_{g,n}(\vec{L}) = 6g-6+2n \overset{!}{=} 2d_1 + \cdots + 2d_n$$

$$\Updownarrow$$

$$3g-3+n = \sum_{i=1}^{n} d_i .$$

Fact. The $\langle \tau \cdots \tau \rangle_g$ are 2D gravity correlators.

Roughly speaking, $\langle \tau \cdots \tau \rangle_g \sim \langle \mathrm{tr}(M^{d_1}) \cdots \mathrm{tr}(M^{d_n}) \rangle$.

Thm. (Duistermaat-Heckman). $\boxed{\omega_{\vec{L}} = \omega + \frac{1}{2}\sum_{i=1}^{n} L_i^2 \psi_i .}$

Cor.
$$\boxed{\begin{aligned} V_{g,n}(\vec{L}) &= \frac{1}{(2\pi^2)^{3g-3+n}(3g-3+n)!} \int_{\overline{\mathcal{M}}_{g,n}(\vec{L})} \left(\omega + \frac{1}{2}\sum_{i=1}^{n} L_i^2 \psi_i\right)^{3g-3+n} \\ &\text{``}=\text{''} \frac{1}{(2\pi^2)^{3g-3+n}} \int_{\overline{\mathcal{M}}_{g,n}(\vec{L})} \exp\left(\omega + \frac{1}{2}\sum_{i=1}^{n} L_i^2 \psi_i\right) \end{aligned}}$$

is a polynomial of degree $6g-6+2n$ in the L_i. The leading term is a linear combo of the $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g$ s.t. the d_i sum to $3g-3+n$. //

$\therefore$ correlators $\equiv$ volumes $\equiv$ int. #s $\equiv$ matrices.

π. 2D Yang-Mills vs. 2D Gravity

2D Yang-Mills	2D Gravity
Σ_g = spacetime $G = SU(2)$, compact	Σ_g = spacetime $G = PSL(2,\mathbb{R})$, non-cpt
M_g = phase space $= \left\{ \text{flat } \mathfrak{g}\text{-valued connections} / \Sigma_g \right\}$	$\mathcal{M}_g$ = moduli space $= \left\{ \text{flat } PSL(2,\mathbb{R}) \text{ connections} / \Sigma_g \right\}$
$\omega = \frac{1}{4\pi}\int_{\Sigma_g} \mathrm{tr}[\delta A \wedge \delta A]$	$\omega \propto \frac{1}{4\pi}\int_{\Sigma_g} \mathrm{tr}[\delta A \wedge \delta A]$
$M_{g,n}$ requires holonomy around k^{th} hole to be conjugate to $\begin{pmatrix} e^{i\alpha_k} & 0 \\ 0 & e^{-i\alpha_k} \end{pmatrix}$	$\mathcal{M}_{g,n}$ requires holonomy around k^{th} hole to be conjugate to $\begin{pmatrix} e^{L_k} & 0 \\ 0 & e^{-L_k} \end{pmatrix}$
$\vec{\alpha} \in (S^1)^n$ compact, Glue pants to get $T_{g,\vec{\alpha}}$	$\vec{L} \in \mathbb{R}_+^n$ non-compact, Glue pants to get $T_{g,\vec{L}}$
$V_{g,\vec{\alpha}} \sim \int d^n\alpha$ finite $\left(\begin{smallmatrix}\text{no}\\ \text{MCG}\end{smallmatrix}\right)$	$V_{g,\vec{L}} \sim \int d^nL$ infinite $\left(\begin{smallmatrix}\text{forgot}\\ \text{MCG}\end{smallmatrix}\right)$

3.2. The MMM Classes [Set $\Gamma = \vec{0}$]

Forms on bundles can be integrated fiberwise.

E.g. Let $Y = M^n_x \times \mathbb{R}_t$ be a trivial $\mathbb{R}$-bundle over M, with projection

$$\pi : Y \longrightarrow M, \quad y = (x,t) \longmapsto x.$$

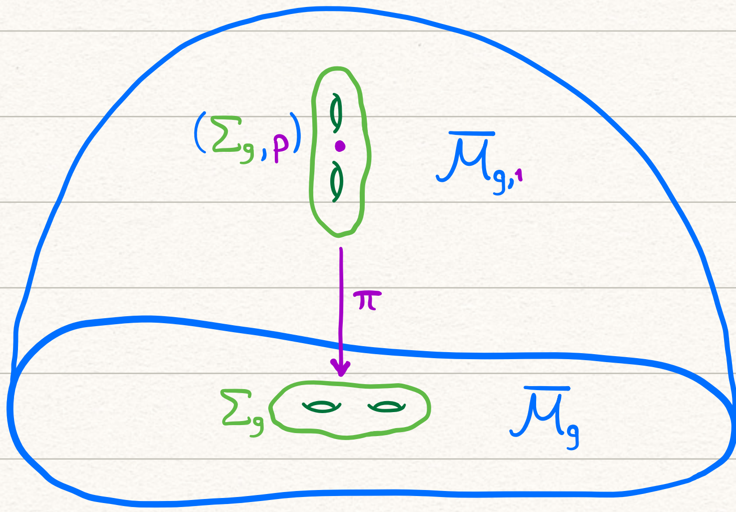
Then $\pi^{-1}(x) = \{x\} \times \mathbb{R}_t$.

Let $\eta = f_{\mu\nu}(x,t)\, dy^\mu dy^\nu = f_{ij}\, dx^i dx^j + f_{it}\, dx^i dt$.

Define $\pi_* \eta = \left[\int_{-\infty}^{\infty} dt\, f_{it}(x,t)\right] dx^i$, a 1-form on M. //

Now $\overline{\mathcal{M}}_{g,1} \sim \overline{\mathcal{M}}_g \times \Sigma_g$ has a "forgetful" map

$$\pi : \overline{\mathcal{M}}_{g,1} \longrightarrow \overline{\mathcal{M}}_g . \quad (\Sigma_g, p) \longmapsto \Sigma_g$$



Consider $\tau_2 = \psi^2 \in H^4(\overline{\mathcal{M}}_{g,1}, \mathbb{Q})$. Integrate over the insertion point $p \in \Sigma_g$ to obtain

$$\boxed{\kappa \equiv \kappa_1 \equiv \pi_* \tau_2 \in H^2(\overline{\mathcal{M}}_g, \mathbb{Q}).}$$

Thm. (spooky). $\boxed{\kappa = \frac{\omega}{2\pi^2}}$ in cohomology. //

[Pedantic aside: this is meaningful because $\omega \in H^\bullet_{\mathbb{Q}}$ and $\kappa \in H^\bullet_{\mathbb{Q}[\pi^2]}$ live in different rings.]

Thus $$V_{g,n}(\vec{0}) = \frac{1}{(2\pi^2)^{3g-3+n}(3g-3+n)!}\int_{\overline{\mathcal{M}}_{g,n}} \omega^{3g-3+n} =$$

$$= \frac{1}{(3g-3+n)!}\int_{\overline{\mathcal{M}}_{g,n}} \kappa^{3g-3+n} = \boxed{\int_{\overline{\mathcal{M}}_{g,n}} e^{\kappa} = V_{g,n}.}$$

Generalize to $\pi : \overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n}$ and define the Miller-Morita-Mumford (MMM) classes

$$\boxed{\kappa_d \equiv \pi_* \tau_{d+1} \in H^{2d}(\overline{\mathcal{M}}_{g,n}).}$$

3.3. Correlators at Last

One hopes to get $\left\langle \tau_{d+1} \prod_{j=1}^{k} \tau_{n_j} \right\rangle_g \overset{?}{=} \left\langle \kappa_d \prod_{j=1}^{k} \tau_{n_j} \right\rangle_g$

by integrating over the τ_{d+1} insertion point.

But the p_i cannot coincide! They collide, generating contact terms: $\tau_{d+1} \cdot \tau_{n_j} \rightsquigarrow \kappa_d + \tau_{d+n_j}$.

E.g. $g=2, n=0$. $\dim(\mathcal{M}_2)=6$, $V_2=\frac{1}{3!}\langle\kappa\kappa\kappa\rangle$.

Consider $\langle\tau_2\tau_2\tau_2\rangle = \langle\kappa\tau_2\tau_2\rangle + 2\langle\tau_3\tau_2\rangle \ldots$

- $\langle\kappa\tau_2\tau_2\rangle = \langle\kappa\kappa\tau_2\rangle + \langle\kappa\tau_3\rangle$
  $= \langle\kappa\kappa\kappa\rangle + \langle\kappa\tau_3\rangle$,

- $\langle\tau_2\tau_3\rangle = \langle\kappa\tau_3\rangle + \langle\tau_4\rangle$.

Thus $\langle\tau_2\tau_2\tau_2\rangle = \langle\kappa\kappa\kappa\rangle + 3\langle\tau_3\tau_2\rangle - \langle\tau_4\rangle$

$$\Updownarrow$$

$$\boxed{\langle\kappa\kappa\kappa\rangle = \langle\tau_2\tau_2\tau_2\rangle - 3\langle\tau_3\tau_2\rangle + \langle\tau_4\rangle.}$$

This allows us to convert between τ & κ.
Indeed, V_2 is the coefficient at ξ^3 in

$\langle\exp(\xi\kappa)\rangle$ **AND** $\langle\exp(\xi\tau_2 - \frac{\xi^2}{2!}\tau_3 + \frac{\xi^3}{3!}\tau_4)\rangle$. //

$\therefore$ For all $g\geq 2$, V_g has a generating $f^{\underline{n}}$:

$$\boxed{\langle\exp(\xi\kappa)\rangle_g = \left\langle\exp\left[\sum_{k=2}^{\infty}\frac{(-1)^k\xi^{k-1}}{(k-1)!}\tau_k\right]\right\rangle_g,}$$ [$\underline{\text{Pf}}$: same as above.]

and V_g is the coefficient at ξ^{3g-3}.

3.4. Comments on Topological Gravity

Previously, we took $\vec{L} = \vec{0}$ & allowed only punctures.
If $\vec{L} \neq \vec{0}$, we immediately get problems:

- $\exists$ boundaries, bulk punctures, bdy punctures.
- $\overline{\mathcal{M}}$ is unorientable and itself has bdy: we cannot define integrals of $H^\bullet$ classes!

Solution: treat this as a gravitational anomaly; cancel it by coupling to matter.

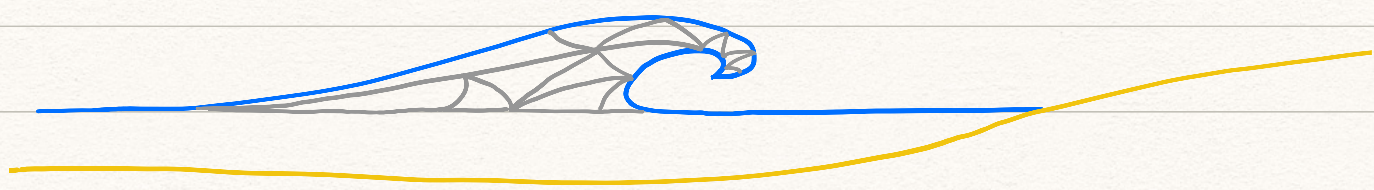
The matter should:

- Be topological
- Have fermions
- Cancel the bdy anomaly.

$\therefore$ Give Σ_g a spin structure, and sum over these.

This theory is related to the Kitaev chain, apparently.

Things get complicated, but the anomalies cancel!



$\boxed{4.}$ Summary & Conclusions

- Riemann surfaces are classified by genus:
  - $g=0$: $\mathbb{P}^1$, $\mathcal{M}_0 = \{*\}$.
  - $g=1$: T^2, $\mathcal{M}_1 = \mathbb{H}^2/SL(2,\mathbb{Z})$.
  - $g\geq 2$: Σ_g, $\mathcal{M}_g = (\mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3})/\Gamma$
- Boundaries: get $\overline{\mathcal{M}}_{g,n}(\vec{L}) \sim \mathbb{R}^{6g-6+2n}$.
- $\overline{\mathcal{M}}_{g,n}(\vec{L})$ is symplectic; $\omega_{\vec{L}} = \sum_{i=1}^{3g-3+n} dl_i \wedge d\theta_i$.
- Volumes $V_{g,n}(\vec{L}) \sim \int_{\overline{\mathcal{M}}_{g,n}} e^{\omega_{\vec{L}}}$ may be computed

  by cutting Σ_g along closed geodesics, using a partition of unity, and recursing.
- DH: $\omega_{\vec{L}} = \omega_{\vec{0}} + \sum_{i=1}^{n} L_i^2 \psi_i \Longrightarrow V_{g,n}(\vec{L})$ polynomial.

  [$\psi_i = c_1(C_i)$; $C_i \xrightarrow{S^1} \overline{\mathcal{M}}_{g,n}$ rotates marked pts.]
- Correlators are intersection numbers: $\tau_{i,d} \equiv \psi_i^d$;

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g \equiv \int_{\overline{\mathcal{M}}_{g,n}(\vec{L})} \tau_{i_1,d_1} \cdots \tau_{i_n,d_n} = \int_{\overline{\mathcal{M}}_{g,n}(\vec{L})} \psi_{i_1}^{d_1} \cdots \psi_{i_n}^{d_n}.$$

- Physics: $\langle \tau \cdots \tau \rangle_g \sim \langle \mathrm{tr}(M^{d_1}) \cdots \mathrm{tr}(M^{d_n}) \rangle$.
- 2D gravity $\equiv$ YM$_2$ w/ gauge gp $PSL(2, \mathbb{R})$.
- $\tau_2 = \psi^2 \in H^4$ can be integrated over the fiber of the forgetful map to yield $\kappa_1 = \kappa \propto \omega \in H^2$.
- Define the MMM classes by $\kappa_d = \pi_* \tau_{d+1} \in H^{2d}$.
- We can relate correlators $\langle \tau \cdots \tau \rangle$ to volumes $\langle \kappa \cdots \kappa \rangle$ by forgetting marked points and introducing contact terms: $\tau_{d+1} \cdot \tau_{n_j} \rightsquigarrow \kappa_d + \tau_{d+n_j}$.
- Result: $\langle \exp(\xi \kappa) \rangle_g = \left\langle \exp\left[\sum_{k=2}^{\infty} \frac{(-1)^k \xi^{k-1}}{(k-1)!} \tau_k \right] \right\rangle_g$

  is a generating $f^{\underline{n}}$ for the WP volumes!
- Topological gravity with boundaries is hard.

→ And in the end it's all just matrices."